① Prove $n-e+f=2$ using induction on e

Base: $P(1)$

 $\Rightarrow$ $n=1$, $e=1$, $f=2$ $\Rightarrow 1-1+2=2$ ✓

 $\Rightarrow$ $n=2$, $e=1$, $f=1$ $\Rightarrow 2-1+1=2$ ✓

I.H.: Assume for some $P(k)$ $k>1$ that the formula holds, $P(k)$ connected

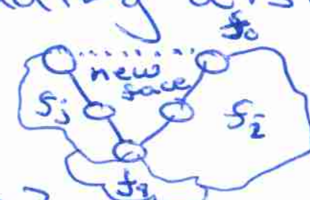
I.S.: Consider graph $G = P(n)$ $n>k$

Case 1: G has no cycles $\Rightarrow$ G is a tree

$n=n$
$e=n-1 \Rightarrow n-(n-1)+1=2$ ✓
$f=1$

Case 2: G has at least one cycle

- Select e from a cycle
- $H=G-e$, I.H. on H
- so $n_H + e_H + f_H = 2$ holds
- Add back the edge to e
- We note that creating a cycle on a planar embedding also creates a face



- $n_G = n_H$
- $e_G = e_H + 1$ $\Rightarrow n_H - e_H + f_H = 2$
- $f_G = f_H + 1$

$n_G-(e_G-1)+(f_G-1)=2$

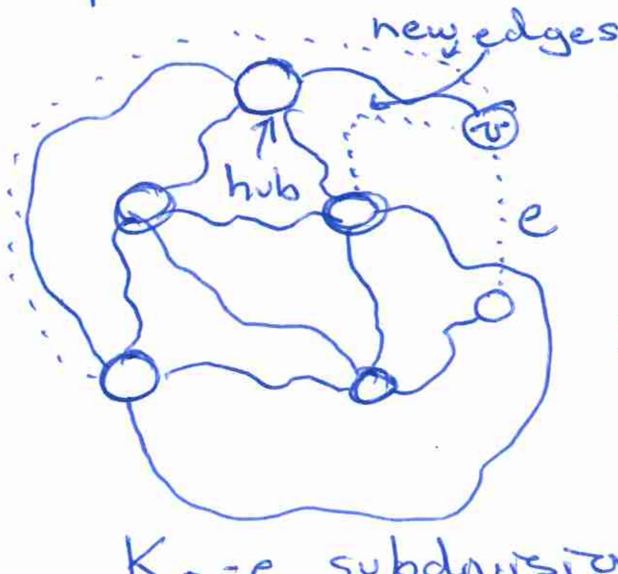
$n_G - e_G + f_G = 2$ □

② G minimal non-planar

$H = G - e$ for some $e \in E(G)$

$\exists e$ s.t. H is maximal planar?

- Removing any edge from G makes the resulting graph planar
- The smallest such nonplanar graphs must necessarily be a K_5 or $K_{3,3}$ subdivision
- For H to be maximally planar, adding any single edge needs to create a K_5 or $K_{3,3}$ subdivision
- On a planar embedding of H, we can naively add edges from a vertex v on a subdivided edge to neighbors of its 'hub' without crossing by following a ~~path~~ route back to the hub then to a neighbor



$K_5 - e$ subdivision

- Hence, G must not have any subdivided edges

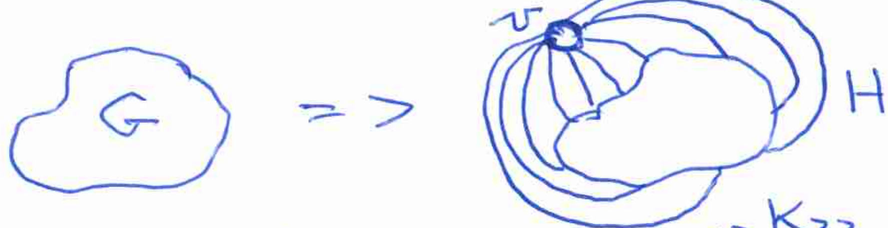
$\Rightarrow$ Only holds when $G = K_5$ $\square$

- On $K_{3,3}$ can still add edges within set

③ G is outerplanar iff it contains no K_4 of $K_{2,3}$ subdivision

($\Rightarrow$) - Let G be outerplanar

- Define $H = G + v$, where v is placed in the outer face of G and attached to all vertices in G



- This cannot create K_5 or $K_{3,3}$ subdivision, as H is planar => so G has no K_4 or $K_{2,3}$ subdivision ✓

($\Leftarrow$) - G has no K_4 or $K_{2,3}$ subdivision and is therefore at least planar

- Construct $H = G + v$ as before, H is also planar, as it has no K_5 or $K_{3,3}$ sub.
- From our proof of Kuratowski's theorem, we know $\exists$ an embedding with v on outer face
- We note that v can reach all $u \in V(G)$ without any edges crossing
- Hence, all $u \in V(G)$ must be on the outer face of an embedding of G

=> G is outerplanar □

(4) Show k-regular graph with cut vertex v must have $\chi'(G) > k$

- We consider that $\chi'(G) = k$
- We note that each edge color will form a matching on G
- As $\chi'(G) = d(v) \; \forall v \in V(G)$ each vertex is an endpoint on all k matchings

$\Rightarrow$ Each edge color forms a perfect match

- Consider H_1 as a component of $G-v$
- We note from $u \in V(H_1)$ there are some $x < k$ vertices in $N(v)$, hence there exists at least one color with a perfect match on $H_1 \Rightarrow |V(H_1)| =$ even
- We make the same argument on the other component of $G-v \Rightarrow H_2$
- So $G-v$ as well as G itself all have perfect matches

$\Rightarrow$ contradiction, as the parity of the ~~edge~~ vertex set can't be even for both $\square$

⑤ Prove for what n on a $4 \times n$ chess board a knight's tour is possible

- We note a "knight's tour" is just a tricky way to say "Hamiltonian cycle ~~Path~~" on graph constructed fom possible movements on the board

- Recall that $c(G-S) \leq |S| \; \forall S \subseteq V(G)$ for a Hamiltonian cycle to exist

- See how we define S below

	x		x		x	
	S		S		S	
S		S		S		
x		x		x		

- Note that each x will be disconnected from $G-S$ as a single vertex

- We have $|S|$ 'x' components plus at least one larger component, so

$$c(G-S) = |S| + 1 \leq |S| \text{ doesn't hold}$$

$\Rightarrow$ No such n allows a tour $\square$